Quick recap on Series

- *n*th Term Test: If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=0}^{\infty} a_k$ diverges
- Geometric Series Test: $\sum_{k=0}^{\infty} r^k$
  - If $|r| \geq 1$, then the series diverges
  - If $|r| < 1$, then the series converges to $\dfrac{1}{1-r}$
- Integral Test: If $a(x) > 0$, decreasing, and $a_k = a(k)$, then

$$\sum_{k=1}^{\infty} a_k \text{ and } \int_1^{\infty} a(x)\,dx$$

behave exactly the same. They either both converge or both diverge

Quick recap on Series (cont.)

- The p-test: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$
  - If $|p| \leq 1$, then the series diverges
  - If $p > 1$, then the series converges

- Direct Comparison Test: If $0 \leq a_k \leq b_k$ then $0 \leq \displaystyle\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k$ and
  - If $\displaystyle\sum_{k=1}^{\infty} b_k$ converges, then $\displaystyle\sum_{k=1}^{\infty} a_k$ also converges
  - If $\displaystyle\sum_{k=1}^{\infty} a_k$ diverges, then $\displaystyle\sum_{k=1}^{\infty} b_k$ also diverges

- Alternating Series Test: If $a_k \geq a_{k+1} > 0$ and $\displaystyle\lim_{k\to\infty} a_k = 0$, then $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges and S_n approximates the series within a_{n+1}

Quick recap on Series (cont.)

- Integral, p, and Direct Comparison tests apply only to positive series
- nth Term, Integral, p, and Direct Comparison determine convergence and/or divergence
  - They do *not* give limit of convergent series
  - Can approximate value by calculating a large partial sum, like S_{100}
- Alternating Series Test determines convergence and error bound when approximating value of convergent alternating series
- Geometric Series Test determines convergence, divergence, *and* value of convergent geometric series

Quick recap on Series (cont.)

- Power Series: A power series in x centered at c is a series

$$P(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots = \sum_{k=0}^{\infty} a_k(x-c)^k$$

where each a_k is a constant. Big idea is that $P(x)$ is a *function*

- Taylor Series: If $f(x)$ is well-behaved at $x = c$, then the Taylor series for $f(x)$ centered at $x = c$ is the series

$$P(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

A Maclaurin series is a Taylor series with $c = 0$: $P(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$

Quick recap on Series (cont.)

We have a catalog of Taylor series we have built

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \qquad = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad = \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \qquad = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(x-1)^k}{k!}$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$