Problem Set #5

Due Friday, October 25, 2024 @ 12:30 pm
Submit as single pdf file to Canvas

*Remember to review the **Guidelines for Problem Sets** on the course webpage when writing up your solutions!*

1. Let $A = \begin{bmatrix} -6 & 24 & 47 \\ 12 & 8 & -45 \\ -12 & -24 & 31 \end{bmatrix}$

   (a) Give a basis for $\text{nul}(A)$ and describe $\text{nul}(A)$ geometrically.

   (b) Give a basis for $\text{col}(A)$ and describe $\text{col}(A)$ geometrically.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

   (a) Find a basis for $\ker(T)$ and describe $\ker(T)$ geometrically.

   (b) Find a basis for $\text{range}(T)$ and describe $\text{range}(T)$ geometrically.

   (c) Describe T in geometric terms.
   e.g. "T rotates the plane by $\frac{\pi}{3}$ radians counter-clockwise", or "T projects the plane onto the x-axis", etc. To be clear, T doesn't do either of these, but these are examples of how you can describe T.
   It may be useful to pick a few specific points in $\mathbb{R}^2$ and see what their image is under T.

3. (a) Let $S = \{\vec{\mathbf{v_1}}, \vec{\mathbf{v_2}}, \ldots, \vec{\mathbf{v_k}}\}$ be a set of vectors in $\mathbb{R}^n$ with $k < n$. Use a theorem from earlier in the semester to explain why S cannot be a basis for $\mathbb{R}^n$.

   (b) Let $S = \{\vec{\mathbf{v_1}}, \vec{\mathbf{v_2}}, \ldots, \vec{\mathbf{v_k}}\}$ be a set of vectors in $\mathbb{R}^n$ with $k > n$. Use a theorem from earlier in the semester to explain why S cannot be a basis for $\mathbb{R}^n$.

   (These problems are essentially the same as Exercises 4.3.29 and 4.3.30 from the text, Lay's Linear Algebra, 4th edition)

4. Let $A = \begin{bmatrix} 3 & 4 & 1 & -1 & 5 \\ 1 & 3 & -2 & 0 & 1 \\ -6 & -8 & -2 & 2 & -10 \\ 5 & 5 & 4 & -2 & 3 \end{bmatrix}$

   (a) Find bases for $\text{col}(A), \text{nul}(A)$, and $\text{row}(A)$.

   (b) What is $\dim \text{nul}(A^T)$? Why?

   (c) One of your answers in (a) is also a basis for $\text{col}(A^T)$. Which one? Why?