Problem Set #3

Due Friday, September 20, 2024 @ 12:30 pm
Submit as single pdf file to Canvas

Remember to review the ***Guidelines for Problem Sets*** *on the course webpage when writing up your solutions!* You may use Mathematica for your calculations unless you are explicitly told to do the computation by hand. When you use Mathematica, be sure to take a screen shot of the relevant parts of your notebook to include in your writeup.

1. Let T be a linear transformation defined by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where $A = \begin{bmatrix} 2 & 4 & 0 \\ -1 & -2 & 9 \\ 2 & 4 & -9 \end{bmatrix}$.

   (a) Let $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. What is $T(\vec{\mathbf{x}})$?

   (b) Let $\vec{\mathbf{b_1}} = \begin{bmatrix} -2 \\ -17 \\ 16 \end{bmatrix}$. Is $\vec{\mathbf{b_1}}$ in the image of T? That is, is there and $\vec{\mathbf{x}}$ where $T(\vec{\mathbf{x}}) = \vec{\mathbf{b_1}}$? If so, is $\vec{\mathbf{x}}$ unique?

   (c) Let $\vec{\mathbf{b_2}} = \begin{bmatrix} 3 \\ 11 \\ -4 \end{bmatrix}$. Is $\vec{\mathbf{b_2}}$ in the image of T? That is, is there and $\vec{\mathbf{x}}$ where $T(\vec{\mathbf{x}}) = \vec{\mathbf{b_2}}$? If so, is $\vec{\mathbf{x}}$ unique?

   (The problem is very similar to Exercises 1.8.3 from the text, Lay's Linear Algebra, 4th edition)

2. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where $A = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & 3 & -2 & 7 \\ 2 & -5 & -9 & 6 \end{bmatrix}$.

   (a) Find all $\vec{\mathbf{x}}$ such that $T(\vec{\mathbf{x}}) = \vec{\mathbf{0}}$.

   (b) Is T one-one? Explain.

   (c) Is T onto $\mathbb{R}^3$? Explain.

3. For each transformation T, find the corresponding matrix A.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ reflects across the line $y = -x$ then rotates by $\frac{\pi}{3}$ radians counter-clockwise about the origin

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates by $\frac{\pi}{3}$ radians counter-clockwise about the origin then reflects across the line $y = -x$

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ rotates about the x-axis counterclockwise by $\frac{\pi}{4}$ radians then projects onto the xy-plane.

4. Let $\vec{\mathbf{v_1}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\vec{\mathbf{v_2}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. If T is a linear transformation such that $T(\vec{\mathbf{v_1}}) = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ and $T(\vec{\mathbf{v_2}}) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, find the corresponding matrix A where $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

   Hint: Write $\vec{\mathbf{e_1}}$ *in terms of* $\vec{\mathbf{v_1}}$ *and* $\vec{\mathbf{v_2}}$. *Then do the same for* $\vec{\mathbf{e_2}}$ *and use that* T *is a linear transformation.*