Let $B = \begin{bmatrix} 75/100 & 15/100 & 5/100 \\ 15/100 & 80/100 & 10/100 \\ 10/100 & 5/100 & 85/100 \end{bmatrix}$

1. For B, find
   (a) The characteristic polynomial
   (b) The eigenvalues
   (c) The corresponding eigenvectors

2. Repeat for ref(B)

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Proof: Suppose $\{\vec{v_1}, \ldots, \vec{v_r}\}$ is a linearly dependent set.
We will show that we get a contradiction.

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- Let $p+1$ be the lowest index of a dependent vector so that $\{\vec{\mathbf{v_1}}, \ldots, \vec{\mathbf{v_p}}\}$ is linearly independent and

$$c_1\vec{\mathbf{v_1}} + c_2\vec{\mathbf{v_2}} + \cdots + c_p\vec{\mathbf{v_p}} = \vec{\mathbf{v_{p+1}}} \quad (*)$$

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- Multiply both sides by λ_{p+1} (we'll see why in a minute)

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- Subtract (**) from (***)

$$c_1(\lambda_1 - \lambda_{p+1})\vec{v_1} + c_2(\lambda_2 - \lambda_{p+1})\vec{v_2} + \cdots + c_p(\lambda_p - \lambda_{p+1})\vec{v_p} = \vec{0}$$

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Therefore, $\{\vec{\mathbf{v_1}}, \ldots, \vec{\mathbf{v_r}}\}$ are linearly independent. $\square$