**THEOREM 9** If a vector space $V$ has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in $V$ containing more than $n$ vectors must be linearly dependent.

**Proof:** Let $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p\}$ be a set in $V$ where $p > n$.
Then $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p\}$ is linearly dependent if there exists a nontrivial solution to

$$x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + \cdots x_p\vec{\mathbf{v}}_p = \vec{\mathbf{0}}$$

*Overview:*

- We will convert this into a matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ where $A$ is $n \times p$.
- Since $p > n$, $A$ has a free variable, and there exists a non-trivial solution to the homogeneous system.
- Thus, $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_p\}$ is a linearly dependent set.
- Note this applies to *any* vector space $V$, not just $\mathbb{R}^n$

## Proof of Theorem 4.9, continued

Since $\mathcal{B}$ is a basis for $V$, we can write

$$\begin{aligned}
a_{11}\vec{\mathbf{b_1}} + a_{12}\vec{\mathbf{b_2}} + \cdots + a_{1n}\vec{\mathbf{b_n}} &= \vec{\mathbf{v_1}} \\
a_{21}\vec{\mathbf{b_1}} + a_{22}\vec{\mathbf{b_2}} + \cdots + a_{2n}\vec{\mathbf{b_n}} &= \vec{\mathbf{v_2}} \\
&\vdots \\
a_{p1}\vec{\mathbf{b_1}} + a_{p2}\vec{\mathbf{b_2}} + \cdots + a_{pn}\vec{\mathbf{b_n}} &= \vec{\mathbf{v_p}}
\end{aligned}$$

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&\vdots \\
a_{p1}\vec{\mathbf{b_1}} + a_{p2}\vec{\mathbf{b_2}} + \cdots + a_{pn}\vec{\mathbf{b_n}} &= \vec{\mathbf{v_p}}
\end{aligned}$$

Remember we are looking for a non-trivial solution to

$$x_1\vec{\mathbf{v_1}} + x_2\vec{\mathbf{v_2}} + \cdots x_p\vec{\mathbf{v_p}} = \vec{\mathbf{0}}$$

Since $\mathcal{B}$ is a basis for $V$, we can write

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Remember we are looking for a non-trivial solution to

$$x_1\vec{\mathbf{v_1}} + x_2\vec{\mathbf{v_2}} + \cdots x_p\vec{\mathbf{v_p}} = \vec{\mathbf{0}}$$

which becomes

$$\begin{aligned} x_1(a_{11}\vec{\mathbf{b_1}} + a_{12}\vec{\mathbf{b_2}} + \cdots + a_{1n}\vec{\mathbf{b_n}}) + \\ x_2(a_{21}\vec{\mathbf{b_1}} + a_{22}\vec{\mathbf{b_2}} + \cdots + a_{2n}\vec{\mathbf{b_n}}) + \\ \cdots + x_p(a_{p1}\vec{\mathbf{b_1}} + a_{p2}\vec{\mathbf{b_2}} + \cdots + a_{pn}\vec{\mathbf{b_n}}) = \vec{\mathbf{0}} \end{aligned}$$

## Proof of Theorem 4.9, continued

We can rearrange

$$\begin{aligned} x_1(a_{11}\vec{\mathbf{b_1}} + a_{12}\vec{\mathbf{b_2}} + \cdots + a_{1n}\vec{\mathbf{b_n}}) + \\ x_2(a_{21}\vec{\mathbf{b_1}} + a_{22}\vec{\mathbf{b_2}} + \cdots + a_{2n}\vec{\mathbf{b_n}}) + \\ \cdots + x_p(a_{p1}\vec{\mathbf{b_1}} + a_{p2}\vec{\mathbf{b_2}} + \cdots + a_{pn}\vec{\mathbf{b_n}}) = \vec{\mathbf{0}} \end{aligned}$$

to

$$\begin{aligned} (x_1a_{11} + x_2a_{21} + \cdots + x_pa_{p1})\vec{\mathbf{b_1}} + \\ (x_1a_{12} + x_2a_{22} + \cdots + x_pa_{p2})\vec{\mathbf{b_2}} + \\ \cdots + (x_1a_{1n} + x_2a_{2n} + \cdots + x_pa_{pn})\vec{\mathbf{b_n}} = \vec{\mathbf{0}} \end{aligned}$$

## Proof of Theorem 4.9, continued

We can rearrange

$$x_1(a_{11}\vec{\mathbf{b_1}} + a_{12}\vec{\mathbf{b_2}} + \cdots + a_{1n}\vec{\mathbf{b_n}}) +$$
$$x_2(a_{21}\vec{\mathbf{b_1}} + a_{22}\vec{\mathbf{b_2}} + \cdots + a_{2n}\vec{\mathbf{b_n}}) +$$
$$\cdots + x_p(a_{p1}\vec{\mathbf{b_1}} + a_{p2}\vec{\mathbf{b_2}} + \cdots + a_{pn}\vec{\mathbf{b_n}}) = \vec{\mathbf{0}}$$

to

$$(x_1a_{11} + x_2a_{21} + \cdots + x_pa_{p1})\vec{\mathbf{b_1}} +$$
$$(x_1a_{12} + x_2a_{22} + \cdots + x_pa_{p2})\vec{\mathbf{b_2}} +$$
$$\cdots + (x_1a_{1n} + x_2a_{2n} + \cdots + x_pa_{pn})\vec{\mathbf{b_n}} = \vec{\mathbf{0}}$$

Remember $\left\{\vec{\mathbf{b_1}}, \ldots, \vec{\mathbf{b_n}}\right\}$ is a linearly independent set.

# Proof of Theorem 4.9, continued

Since $\left\{ \vec{\mathbf{b_1}}, \ldots, \vec{\mathbf{b_n}} \right\}$ is a linearly independent, we get

$$\begin{aligned} x_1 a_{11} + x_2 a_{21} + \cdots + x_p a_{p1} &= 0 \\ x_1 a_{12} + x_2 a_{22} + \cdots + x_p a_{p2} &= 0 \\ &\vdots \\ x_1 a_{1n} + x_2 a_{2n} + \cdots + x_p a_{pn} &= 0 \end{aligned}$$

# Proof of Theorem 4.9, continued

Since $\left\{\vec{\mathbf{b}_1}, \ldots, \vec{\mathbf{b}_n}\right\}$ is a linearly independent, we get

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This converts to the matrix equation

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{p1} \\ a_{12} & a_{22} & \cdots & a_{p2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \vec{\mathbf{0}}$$

# Proof of Theorem 4.9, continued

The matrix equation

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{p1} \\ a_{12} & a_{22} & \cdots & a_{p2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \vec{\mathbf{0}}$$

is the same as $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ where $A$ is $n \times p$ with $p > n$.

# Proof of Theorem 4.9, continued

The matrix equation

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{p1} \\ a_{12} & a_{22} & \cdots & a_{p2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \vec{\mathbf{0}}$$

is the same as $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ where $A$ is $n \times p$ with $p > n$.

Thus, $A$ has a free variable and $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a non-trivial solution.

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This gives us a non-trivial solution to

$$x_1\vec{\mathbf{v_1}} + x_2\vec{\mathbf{v_2}} + \cdots x_p\vec{\mathbf{v_p}} = \vec{\mathbf{0}}$$

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This gives us a non-trivial solution to

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Thus, $\{\vec{\mathbf{v_1}}, \ldots, \vec{\mathbf{v_p}}\}$ must be linearly dependent. $\square$

# Give the dimension of each vector space

1. $\mathbb{R}^2$
2. $\mathbb{R}^5$
3. $\mathbb{R}^n$
4. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & -2 & 3 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
   (a) $\text{col}(A)$
   (b) $\text{nul}(A)$
   (c) $\text{row}(A)$

1. Let $A = \begin{bmatrix} 1 & 24 & -13 & -12 \\ 1 & 3 & -2 & -1 \\ 7 & 0 & -3 & 4 \end{bmatrix}$. Find bases for col($A$), nul($A$), and row($A$).

2. If $A$ is $6 \times 11$ of rank 4, what is the dimension of nul($A$)?

3. If $A$ is the matrix corresponding to a one-one linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^8$, what is the dimension of nul($A$)? of row($A$)? of nul($A^T$)?