THEOREM 5

The Spanning Set Theorem

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

From Lay, Section 4.3

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👋 Justification:

a. We are given $\vec{\mathbf{v_k}} = a_1\vec{\mathbf{v_1}} + a_2\vec{\mathbf{v_2}} + \cdots + a_{k-1}\vec{\mathbf{v_{k-1}}} + a_{k+1}\vec{\mathbf{v_{k+1}}} + \cdots + a_p\vec{\mathbf{v_p}}$

Let $\vec{\mathbf{u}} \in \text{Span}\left\{\vec{\mathbf{v_1}}, \dots, \vec{\mathbf{v_p}}\right\}$ so that

$$\begin{aligned}\vec{\mathbf{u}} &= c_1\vec{\mathbf{c_1}} + \cdots + c_k\vec{\mathbf{v_k}} + \cdots + c_p\vec{\mathbf{v_p}} \\ &= c_1\vec{\mathbf{c_1}} + \cdots + c_k\left(a_1\vec{\mathbf{v_1}} + a_2\vec{\mathbf{v_2}} + \cdots + a_{k-1}\vec{\mathbf{v_{k-1}}} + a_{k+1}\vec{\mathbf{v_{k+1}}} + \cdots + a_p\vec{\mathbf{v_p}}\right) + \cdots + c_p\vec{\mathbf{v_p}}\end{aligned}$$

Thus, $\vec{\mathbf{u}} \in \text{Span}\left\{\vec{\mathbf{v_1}}, \dots, \vec{\mathbf{v_{k-1}}}, \vec{\mathbf{v_{k+1}}}, \vec{\mathbf{v_p}}\right\} \quad \Rightarrow \quad \text{Span } S = \text{Span }\{S - \vec{\mathbf{v_k}}\}$

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a. If one of the vectors in S—say, $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

👋 Justification:

b. We just keep throwing out vectors until we have a linearly independent set.

We haven't changed the span, so the remaining set must be a basis.

The condition $H \neq \left\{\vec{\mathbf{0}}\right\}$ is a technicality since the subspace $H = \left\{\vec{\mathbf{0}}\right\}$ has no basis:

$H = \text{Span}\left\{\vec{\mathbf{0}}\right\}$, but $\left\{\vec{\mathbf{0}}\right\}$ is not a linearly independent set

1. Find a basis for nul(A) and for col(A) if $A = \begin{bmatrix} 1 & 2 & -7 & 4 & 4 \\ -2 & 3 & -21 & 13 & 2 \\ 4 & -7 & 47 & -29 & 1 \\ -2 & -3 & 9 & -5 & 4 \end{bmatrix}$

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that performs a perpendicular projection onto the line $y = 2x$

   (a) Find a basis for $\ker(T)$

   (b) Find a basis for range(T)

   Do your answers make sense geometrically?