Some Big Ideas, Week 12

Apr 17 – Apr 21, 2023

- **Definition**: A **graph** $G = (V, E)$ consists of a non-empty set V, called the **vertices**, and a set E, called the **edges**, of two element subsets of V.
  - Two vertices v and w in G are **adjacent** if $\{v, w\}$ is an edge in G.
  - Note that we allow the possibility of more than one edge from one vertex to another, and we can have loops where an edge begins and ends at the same vertex.
  - In a *directed graph*, or digraph, the edges are ordered pairs. cf. Problem Set #7, question 4.
- **Definition**: An **isomorphism** between graphs G_1 and G_2 is a bijection $f : V_1 \longrightarrow V_2$ between the vertices of the graphs such that $\{a, b\}$ is an edge in G_1 iff $\{f(a), f(b)\}$ is an edge in G_2.
- **Definition**: A **simple graph** is a graph without any loops or duplicate edges between two vertices.

  For any $n \in \mathbb{N}$, the **complete graph on n vertices**, denoted K_n, is a simple graph with n vertices with exactly one edge connecting each pair of distinct vertices.
- **Definition**: The **degree** of a vertex v of a graph G, denoted $\deg(v)$, is the number of edges with v as an endpoint. The **total degree** of G is the sum of all the degrees of all of the vertices of G.
- **The Handshake Lemma:** In any graph, the total degree is twice the number of edges.
- **Definitions**: Let G be a graph and v and w vertices in G.
  - A **walk from v to w** is a sequence of vertices $v = v_0 v_1 v_2 \ldots v_k = w$ such that v_i is adjacent to v_{i+1} for $i = 0, \ldots, k-1$.
  - A **trail from v to w** is a walk from v to w without a repeated edge.
  - A **path from v to w** is a trail from v to w without a repeated vertex.
    Note a path is a walk with no repeated edges or no repeated vertices.
  - A **closed walk** is a walk that starts and ends at the same vertex.
  - A **circuit** is a closed walk with at least one edge and with no repeated edges.
  - A **simple circuit** is a circuit with no repeated vertex except for the beginning and ending vertex.
  - An **Euler circuit** for a graph G is a circuit that contains every vertex and every edge of G.
  - A **Hamiltonian circuit** for a graph G is a simple circuit that includes every vertex of G.
- **Definition**: A graph G is **connected** iff for all vertices v and w in G, there is a walk from v to w.
- Levin, Discrete Mathematics, An Open Introduction, 3rd edition, gives a good summary of definitions on pages 242 & 243.

Some of the resources I used in constructing the Big Ideas notes this semester are: Ernst: *Introduction to Proof via Inquiry-Based Learning*; Epp: *Discrete Mathematics with Applications, 4th edition*; Levin: *Discrete Mathematics, An Open Introduction, 3rd edition*; Sundstrom: *Mathematical Reasoning, Writing and Proof, Version 3*; and the notes of my colleague, Rachelle DeCoste at Wheaton.

Check the Tentative Weekly Syllabus for the specific sections relevant for this week.