Problem Set #6

Due Thursday, November 2, 2023 @ 11:59 pm
Submit as single pdf file to Canvas

Remember to review the Guidelines for Problem Sets on the course webpage when writing up the solutions with your group, and don't forget to submit the Partner Evaluation through Canvas.

1. Let $\mathcal{B} = \left\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3\right\}$ where $\vec{\mathbf{b}}_1 = \begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix}, \vec{\mathbf{b}}_2 = \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix}, \vec{\mathbf{b}}_3 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$

   (a) Show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

   (b) Find $[\vec{\mathbf{x}}]_{\mathcal{B}}$, the coordinates for $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ relative to the basis $\mathcal{B}$.

   (c) Let $P_{\mathcal{B}} = [\vec{\mathbf{b}}_1\ \vec{\mathbf{b}}_2\ \vec{\mathbf{b}}_3]$, the matrix whose columns are the basis $\mathcal{B}$.
   This matrix is called the *change-of-coordinates* matrix from $\mathcal{B}$ to the standard basis for $\mathbb{R}^3$.
   Using the values from (b), verify that $P_{\mathcal{B}}[\vec{\mathbf{x}}]_{\mathcal{B}} = \vec{\mathbf{x}}$.

   (d) If $[\vec{\mathbf{u}}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 63 \\ 76 \end{bmatrix}$, use $P_{\mathcal{B}}$ to find $\vec{\mathbf{u}}$.

   (e) If $\vec{\mathbf{v}} = \begin{bmatrix} 5 \\ 4 \\ -7 \end{bmatrix}$, use $P_{\mathcal{B}}^{-1}$ to find $[\vec{\mathbf{v}}]_{\mathcal{B}}$.

2. Let $A = \begin{bmatrix} 3 & 4 & 1 & -1 & 5 \\ 1 & 3 & -2 & 0 & 1 \\ -6 & -8 & -2 & 2 & -10 \\ 5 & 5 & 4 & -2 & 3 \end{bmatrix}$

   (a) Find bases for $\text{col}(A), \text{nul}(A)$, and $\text{row}(A)$.

   (b) What is $\dim \text{nul}(A^T)$? Why?

   (c) One of your answers in (a) is also a basis for $\text{col}(A^T)$. Which one? Why?

3. Suppose A is the matrix corresponding to an onto linear transformation $T : \mathbb{R}^7 \to \mathbb{R}^3$.

   (a) What is the dimension of $\text{nul}(A)$? $\text{col}(A)$? Why?

   (b) What is $\text{range}(T)$? Why?

   (c) Describe $\text{col}(A^T)$ geometrically.

4. Let $P = \begin{bmatrix} 0.3 & 0.1 & 0 \\ 0.2 & 0 & 0.8 \\ 0.5 & 0.9 & 0.2 \end{bmatrix}$

   (a) Show that P is a regular stochastic matrix.

   (b) Find the steady-state vector for P.