Problem Set #3

Due Thursday, September 21, 2023 @ 11:59 pm
Submit as single pdf file to Canvas

Remember to review the Guidelines for Problem Sets on the course webpage when writing up the solutions with your group, and don't forget to submit the Partner Evaluation through Canvas.

1. Let T be a linear transformation defined by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where $A = \begin{bmatrix} 2 & 4 & 0 \\ -1 & -2 & 9 \\ 2 & 4 & -9 \end{bmatrix}$.

   (a) Let $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$. What is $T(\vec{\mathbf{x}})$?

   (b) Let $\vec{\mathbf{b}}_1 = \begin{bmatrix} -2 \\ -17 \\ 16 \end{bmatrix}$. Is $\vec{\mathbf{b}}_1$ in the image of T? That is, is there and $\vec{\mathbf{x}}$ where $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}_1$? If so, is $\vec{\mathbf{x}}$ unique?

   (c) Let $\vec{\mathbf{b}}_2 = \begin{bmatrix} 3 \\ 11 \\ -4 \end{bmatrix}$. Is $\vec{\mathbf{b}}_2$ in the image of T? That is, is there and $\vec{\mathbf{x}}$ where $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}_2$? If so, is $\vec{\mathbf{x}}$ unique?

   (The problem is very similar to Exercises 1.8.3 from the text, Lay's Linear Algebra, 4th edition)

2. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where $A = \begin{bmatrix} 1 & 2 & 4 & -1 \\ 0 & 3 & -2 & 7 \\ 2 & -5 & -9 & 6 \end{bmatrix}$.

   (a) Find all $\vec{\mathbf{x}}$ such that $T(\vec{\mathbf{x}}) = \vec{\mathbf{0}}$.

   (b) Is T one-one? Explain.

   (c) Is T onto $\mathbb{R}^3$? Explain.

3. For each transformation T, find the corresponding matrix A.

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^2$ reflects across the line $y = -x$ then rotates by $\frac{\pi}{3}$ radians counter-clockwise about the origin

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates by $\frac{\pi}{3}$ radians counter-clockwise about the origin then reflects across the line $y = -x$

   (c) $T : \mathbb{R}^3 \to \mathbb{R}^3$ rotates about the x-axis counterclockwise by $\frac{\pi}{4}$ radians then projects onto the xy-plane.

4. Let $\vec{\mathbf{v}}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\vec{\mathbf{v}}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. If T is a linear transformation such that $T(\vec{\mathbf{v}}_1) = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$ and $T(\vec{\mathbf{v}}_2) = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, find the corresponding matrix A where $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

   Hint: Write $\vec{\mathbf{e}}_1$ in terms of $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_2$. Then do the same for $\vec{\mathbf{e}}_2$ and use that T is a linear transformation.