## The Invertible Matrix Theorem (Theorem 2.8)

Let *A* be a square  $n \times n$  matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

- a. A is an invertible matrix.
- b. *A* is row equivalent to the  $n \times n$  identity matrix.
- c. A has n pivot positions.
- d. The equation  $A\vec{x} = \vec{0}$  has only the trivial solution.
- e. The columns of *A* form a linearly independent set.
- f. The linear transformation  $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$  is one-to-one.
- g. The equation  $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ . has at least one solution for each  $\vec{\mathbf{b}}$  in  $\mathbb{R}^n$ .
- h. The columns of A span  $\mathbb{R}^n$ .
- i. The linear transformation  $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$  maps  $\mathbb{R}^n$  onto  $\mathbb{R}^n$ .
- j. There is an  $n \times n$  matrix C such that CA = I.
- k. There is an  $n \times n$  matrix D such that AD = I.
- l.  $A^T$  is an invertible matrix.

Recall Theorems 1.4 and 1.12:

## Theorem 1.4

Let *A* be an  $m \times n$  matrix. The following are equivalent:

- a. For all  $\vec{\mathbf{b}} \in \mathbb{R}^m$ ,  $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$  has a solution.
- b. Each  $\vec{\mathbf{b}} \in \mathbb{R}^m$  is a linear combination of the columns of A.
- c. The columns of *A* span  $\mathbb{R}^m$
- d. A has a pivot in every row.

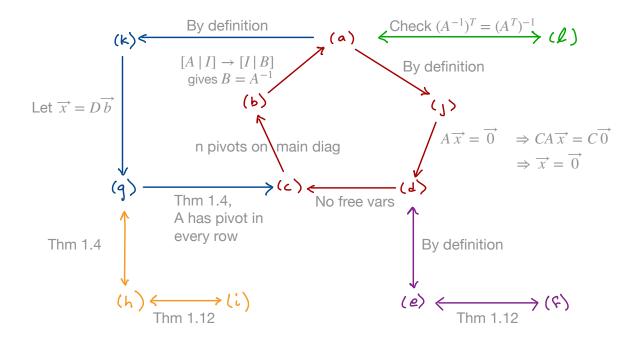
## Theorem 1.12

Let  $T: \mathbb{R}^n \to \mathbb{R}^m$  be a linear transformation and A the standard matrix for T.

- a. T maps  $\mathbb{R}^n$  onto  $\mathbb{R}^m$  iff the columns of A span  $\mathbb{R}^m$
- b. T is one-one iff the columns of A are linearly independent

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## Sketch of proof of the Invertible Matrix Theorem



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