From Lay, Section 4.4

THEOREM 7

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n \tag{1}$$

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- Then subtracting gives
  $$\vec{\mathbf{0}} = (c_1 - d_1)\vec{\mathbf{b_1}} + \cdots + (c_n - d_n)\vec{\mathbf{b_n}}$$
- Since $\mathcal{B}$ is a linearly independent set, we know that
  $$c_1 = d_1, c_2 = d_2, \dots, c_n = d_n \; \square$$

Give the dimension of each vector space

1. $\mathbb{R}^5$
2. $\mathbb{R}^n$
3. $\mathbb{P}_2$
4. $\mathbb{P}_n$
5. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & -2 & 3 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$
   (a) $\text{col}(A)$
   (b) $\text{nul}(A)$
   (c) $\text{row}(A)$

THEOREM 9 If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Proof: Let $\{\vec{\mathbf{v_1}}, \dots, \vec{\mathbf{v_p}}\}$ be a set in V where $p > n$.
Then $\{\vec{\mathbf{v_1}}, \dots, \vec{\mathbf{v_p}}\}$ is linearly dependent if there exists a nontrivial solution to

$$x_1\vec{\mathbf{v_1}} + x_2\vec{\mathbf{v_2}} + \cdots x_p\vec{\mathbf{v_p}} = \vec{\mathbf{0}}$$

<u>*Overview:*</u>

- We will convert this into a matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ where A is $n \times p$.
- Since $p > n$, A has a free variable, and there exists a non-trivial solution to the homogeneous system.
- Thus, $\{\vec{\mathbf{v_1}}, \dots, \vec{\mathbf{v_p}}\}$ is a linearly dependent set.
- Note this applies to *any* vector space V, not just $\mathbb{R}^n$

Proof of Theorem 4.9, continued

Since $\mathcal{B}$ is a basis for V, we can write

$$
\begin{aligned}
a_{11}\vec{\mathbf{b_1}} + a_{12}\vec{\mathbf{b_2}} + \cdots + a_{1n}\vec{\mathbf{b_n}} &= \vec{\mathbf{v_1}} \\
a_{21}\vec{\mathbf{b_1}} + a_{22}\vec{\mathbf{b_2}} + \cdots + a_{2n}\vec{\mathbf{b_n}} &= \vec{\mathbf{v_2}} \\
&\vdots \\
a_{p1}\vec{\mathbf{b_1}} + a_{p2}\vec{\mathbf{b_2}} + \cdots + a_{pn}\vec{\mathbf{b_n}} &= \vec{\mathbf{v_p}}
\end{aligned}
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\end{aligned}$$

Remember we are looking for a non-trivial solution to

$$x_1\vec{\mathbf{v_1}} + x_2\vec{\mathbf{v_2}} + \cdots x_p\vec{\mathbf{v_p}} = \vec{\mathbf{0}}$$

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Remember we are looking for a non-trivial solution to

$$x_1\vec{\mathbf{v_1}} + x_2\vec{\mathbf{v_2}} + \cdots x_p\vec{\mathbf{v_p}} = \vec{\mathbf{0}}$$

which becomes

$$\begin{aligned} x_1(a_{11}\vec{\mathbf{b_1}} + a_{12}\vec{\mathbf{b_2}} + \cdots + a_{1n}\vec{\mathbf{b_n}})+ \\ x_2(a_{21}\vec{\mathbf{b_1}} + a_{22}\vec{\mathbf{b_2}} + \cdots + a_{2n}\vec{\mathbf{b_n}})+ \\ \cdots + x_p(a_{p1}\vec{\mathbf{b_1}} + a_{p2}\vec{\mathbf{b_2}} + \cdots + a_{pn}\vec{\mathbf{b_n}}) = \vec{\mathbf{0}} \end{aligned}$$

We can rearrange

$$\begin{aligned}x_1(a_{11}\vec{\mathbf{b_1}} + a_{12}\vec{\mathbf{b_2}} + \cdots + a_{1n}\vec{\mathbf{b_n}})+\\ x_2(a_{21}\vec{\mathbf{b_1}} + a_{22}\vec{\mathbf{b_2}} + \cdots + a_{2n}\vec{\mathbf{b_n}})+\\ \cdots + x_p(a_{p1}\vec{\mathbf{b_1}} + a_{p2}\vec{\mathbf{b_2}} + \cdots + a_{pn}\vec{\mathbf{b_n}}) = \vec{\mathbf{0}}\end{aligned}$$

to

$$\begin{aligned}(x_1a_{11} + x_2a_{21} + \cdots + x_pa_{p1})\vec{\mathbf{b_1}}+\\ (x_1a_{12} + x_2a_{22} + \cdots + x_pa_{p2})\vec{\mathbf{b_2}}+\\ \cdots + (x_1a_{1n} + x_2a_{2n} + \cdots + x_pa_{pn})\vec{\mathbf{b_n}} = \vec{\mathbf{0}}\end{aligned}$$

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to

$$\begin{gathered} (x_1a_{11} + x_2a_{21} + \cdots + x_pa_{p1})\vec{\mathbf{b_1}} + \\ (x_1a_{12} + x_2a_{22} + \cdots + x_pa_{p2})\vec{\mathbf{b_2}} + \\ \cdots + (x_1a_{1n} + x_2a_{2n} + \cdots + x_pa_{pn})\vec{\mathbf{b_n}} = \vec{\mathbf{0}} \end{gathered}$$

Remember $\left\{\vec{\mathbf{b_1}}, \ldots, \vec{\mathbf{b_n}}\right\}$ is a linearly independent set.

Proof of Theorem 4.9, continued

Since $\left\{\vec{\mathbf{b}_1}, \ldots, \vec{\mathbf{b}_n}\right\}$ is a linearly independent, we get

$$\begin{aligned} x_1 a_{11} + x_2 a_{21} + \cdots + x_p a_{p1} &= 0 \\ x_1 a_{12} + x_2 a_{22} + \cdots + x_p a_{p2} &= 0 \\ &\vdots \\ x_1 a_{1n} + x_2 a_{2n} + \cdots + x_p a_{pn} &= 0 \end{aligned}$$

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This converts to the matrix equation

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{p1} \\ a_{12} & a_{22} & \cdots & a_{p2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \vec{\mathbf{0}}$$

Proof of Theorem 4.9, continued

The matrix equation

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is the same as $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ where A is $n \times p$ with $p > n$.

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Thus, A has a free variable and $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a non-trivial solution.

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This gives us a non-trivial solution to

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Thus, $\{\vec{\mathbf{v_1}}, \ldots, \vec{\mathbf{v_p}}\}$ must be linearly dependent. □