Definition:
For each $n \in \mathbb{N}$, let f_n be a function with domain $A \subset \mathbb{R}$.

The sequence of functions (f_n) **converges pointwise** to a function $f : A \to \mathbb{R}$ iff for all $x \in A$, the sequence $f_n(x)$ converges to $f(x)$.

Definition:
For each $n \in \mathbb{N}$, let f_n be a function with domain $A \subset \mathbb{R}$.

The sequence of functions (f_n) **converges uniformly** to a function $f : A \to \mathbb{R}$ iff for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ and $x \in A$ implies that $|f_n(x) - f(x)| < \epsilon$.

In other words, we can make $|f_n(x) - f(x)|$ small independent of the $x \in A$ chosen.

Theorem 6.2.5 (Cauchy Criterion for Uniform Convergence):
A sequence of functions (f_n) defined on a set $A \subset \mathbb{R}$ converges uniformly on A iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \epsilon$ whenever $m, n \geq N$ and $x \in A$.

This basically says each $(f_n(x))$ is a Cauchy sequence and our choice of N is independent of x.

Theorem 6.2.6:
Let (f_n) be a sequence of functions defined on $A \subset \mathbb{R}$ that converges uniformly on A to a function f.
If each f_n is continuous at $c \in A$, then f is continuous at c.

Theorem 6.3.1:
Suppose (f_n) converges to f pointwise on the closed interval $[a, b]$ and that each f_n is differentiable on $[a, b]$.

If (f_n') converges uniformly on $[a, b]$ to g, then f is differentiable and $f' = g$.

Theorem 6.4.2:
If each f_n is continuous on A and $\sum f_n$ converges uniformly on A to f, then f is continuous on A.

Theorem 6.4.3:
If $f(x) = \sum f_n(x)$ and $\sum f_n'(x)$ converges uniformly, then $f'(x) = \sum f_n'(x)$.

Theorem 6.4.4 (Cauchy Criterion):
A series $\sum f_n$ converges uniformly on $A \subset \mathbb{R}$ iff for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > m \geq N$ implies

$$|s_n(x) - s_m(x)| = |f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \epsilon \text{ for all } x \in A$$

Corollary 6.4.5 (Weierstrass M-Test):
For each $n \in \mathbb{N}$, let f_n be a function defined on $A \subset \mathbb{R}$, and let $M_n \in \mathbb{R}$ be positive such that

$$|f_n(x)| \leq M_n \text{ for all } x \in A$$

If $\sum M_n$ converges, then $\sum f_n$ converges uniformly on A.

Can use to show continuity of continuous everywhere, differentiable nowhere function.