Theorem 4.3.4 (Algebraic Continuity Theorem): Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at $c \in A$. Then

(i) $kf(x)$ is continuous for all $k \in \mathbb{R}$

(ii) $f(x) + g(x)$ is continuous at c

(iii) $f(x)g(x)$ is continuous at c

(iv) $f(x)/g(x)$ is continuous at c if the quotient is defined

Theorem 4.3.9: Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ where $f(A) \subset B$. If f is continuous at c and g is continuous at $f(c)$, then $f \circ g$ is continuous at c.

Theorem 4.4.1: Let $f : A \to \mathbb{R}$ be continuous and let $K \subset A$ be compact. Then $f(K)$ is compact.

Theorem 4.4.2 (Extreme Value Theorem): If $f : K \to \mathbb{R}$ is continuous on the compact set K, then f attains a maximum and minimum value on K.

That is, there exist $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.

Definition 4.4.4: A function $f : A \to \mathbb{R}$ is **uniformly continuous** on A iff for all $\epsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \epsilon$.

Theorem 4.4.5: A function $f : A \to \mathbb{R}$ is not *uniformly* continuous on A iff there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A such that

$$\lim |x_n - y_n| = 0 \qquad \text{but} \qquad |f(x_n) - f(y_n)| \geq \epsilon_0 \text{ for all } n$$

Theorem 4.4.7: If $f : K \to \mathbb{R}$ is continuous and K is compact, then f is uniformly continuous on K.

1. Let $f : A \to \mathbb{R}$ be continuous on A
   (a) If $B \subset A$ is open, is $f(B)$ open? Consider $f(x) = x^2,\ B = (-1, 1)$
   (b) If $B \subset A$ is closed, is $f(B)$ closed? Consider $f(x) = \frac{1}{x},\ B = [1, \infty)$
   (c) If $B \subset A$ is bounded, is $f(B)$ bounded? Consider $f(x) = \frac{1}{x},\ B = (0, 1)$
   (d) If $B \subset A$ is compact, is $f(B)$ compact?
   (e) Show that the Extreme Value Theorem does *not* hold if A is not compact.

2. Show that $f(x) = 3x + 1$ is uniformly continuous on $\mathbb{R}$.

3. Use Theorem 4.4.5 to show that $g(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.
   Hint: Let $(x_n) = \left(n + \frac{1}{n}\right)$ and $(y_n) = \left(n - \frac{1}{n}\right)$