Problem Set #6

Due Thursday, November 4, 2021 @ midnight
Submit as single pdf file to onCourse

Remember to review the Guidelines for Problem Sets on the course webpage.

1. Let $\mathcal{B} = \left\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3\right\}$ where $\vec{\mathbf{b}}_1 = \begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix}, \vec{\mathbf{b}}_2 = \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix}, \vec{\mathbf{b}}_3 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$

   (a) Show that $\mathcal{B}$ is a basis for $\mathbb{R}^3$.

   (b) Find $[\vec{\mathbf{x}}]_{\mathcal{B}}$, the coordinates for $\vec{\mathbf{x}} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ relative to the basis $\mathcal{B}$.

   (c) Let $P_{\mathcal{B}} = [\vec{\mathbf{b}}_1\ \vec{\mathbf{b}}_2\ \vec{\mathbf{b}}_3]$, the matrix whose columns are the basis $\mathcal{B}$.
   This matrix is called the *change-of-coordinates* matrix from $\mathcal{B}$ to the standard basis for $\mathbb{R}^3$.
   Using the values from (b), verify that $P_{\mathcal{B}}[\vec{\mathbf{x}}]_{\mathcal{B}} = \vec{\mathbf{x}}$.

   (d) If $[\vec{\mathbf{u}}]_{\mathcal{B}} = \begin{bmatrix} -4 \\ 63 \\ 76 \end{bmatrix}$, use $P_{\mathcal{B}}$ to find $\vec{\mathbf{u}}$.

   (e) If $\vec{\mathbf{v}} = \begin{bmatrix} 5 \\ 4 \\ -7 \end{bmatrix}$, use $P_{\mathcal{B}}^{-1}$ to find $[\vec{\mathbf{v}}]_{\mathcal{B}}$.

2. Let $A = \begin{bmatrix} 3 & 4 & 1 & -1 & 5 \\ 1 & 3 & -2 & 0 & 1 \\ -6 & -8 & -2 & 2 & -10 \\ 5 & 5 & 4 & -2 & 3 \end{bmatrix}$

   (a) Find bases for $\text{col}(A)$, $\text{nul}(A)$, and $\text{row}(A)$.

   (b) What is $\dim \text{nul}(A^T)$? Why?

   (c) One of your answers in (a) is also a basis for $\text{col}(A^T)$. Which one? Why?

3. Suppose A is the matrix corresponding to an onto linear transformation $T : \mathbb{R}^7 \to \mathbb{R}^3$.

   (a) What is the dimension of $\text{nul}(A)$? $\text{col}(A)$? Why?

   (b) What is $\text{range}(T)$? Why?

   (c) Describe $\text{col}(A^T)$ geometrically.