The Invertible Matrix Theorem (Theorem 2.8)

Let *A* be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

- a. *A* is an invertible matrix.
- b. *A* is row equivalent to the $n \times n$ identity matrix.
- c. *A* has *n* pivot positions.
- d. The equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has only the trivial solution.
- e. The columns of A form a linearly independent set.
- f. The linear transformation $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ is one-to-one.
- g. The equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$. has at least one solution for each $\vec{\mathbf{b}}$ in \mathbb{R}^n .
- h. The columns of *A* span \mathbb{R}^n .
- i. The linear transformation $\vec{\mathbf{x}} \to A\vec{\mathbf{x}}$ maps \mathbb{R}^n onto \mathbb{R}^n .
- j. There is an $n \times n$ matrix *C* such that CA = I.
- k. There is an $n \times n$ matrix *D* such that AD = I.
- l. A^T is an invertible matrix.

Recall Theorems 1.4 and 1.12:

Theorem 1.4

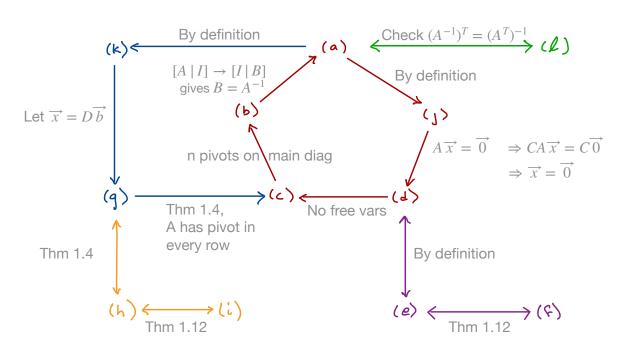
Let *A* be an $m \times n$ matrix. The following are equivalent:

- a. For all $\vec{\mathbf{b}} \in \mathbb{R}^m$, $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a solution.
- b. Each $\vec{\mathbf{b}} \in \mathbb{R}^m$ is a linear combination of the columns of *A*.
- c. The columns of A span \mathbb{R}^m
- d. A has a pivot in every row.

Theorem 1.12

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and *A* the standard matrix for *T*.

- a. T maps \mathbb{R}^n onto \mathbb{R}^m iff the columns of A span \mathbb{R}^m
- b. T is one-one iff the columns of A are linearly independent



Sketch of proof of the Invertible Matrix Theorem