From Lay, Section 1.4

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

Answer True / False

Let $\begin{aligned} \vec{\mathbf{v_1}} &= \langle 1, 3, 18, 2 \rangle \\ \vec{\mathbf{v_2}} &= \langle 2, -1, 9, 0 \rangle \\ \vec{\mathbf{v_3}} &= \langle 3, 2, -4, 1 \rangle \\ \vec{\mathbf{v_4}} &= \langle 4, 7, 1, 3 \rangle \end{aligned}$ and $A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 1 & 4 \\ 6 & 2 & 1 \\ 5 & -17 & 32 \end{bmatrix}$

1. The columns of A span $\mathbb{R}^4$
2. The vectors $\{\vec{\mathbf{v_1}}, \vec{\mathbf{v_2}}, \vec{\mathbf{v_3}}, \vec{\mathbf{v_4}}\}$ span $\mathbb{R}^4$
3. Let $B = [\vec{\mathbf{v_1}}\ \vec{\mathbf{v_2}}\ \vec{\mathbf{v_3}}\ \vec{\mathbf{v_4}}]$ and $\vec{\mathbf{b}} = \langle 72, -128, \pi, e^{-411} \rangle$
   The matrix equation $B\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has a unique solution
4. There exists $\vec{\mathbf{b}} \in \mathbb{R}^4$ such that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has infinitely many solutions.

5. Let $A = \begin{bmatrix} 1 & 3 & 5 \\ -2 & -6 & 7 \end{bmatrix}$

   (a) Find all solutions to the homogeneous system $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

   (b) Find all solutions to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where $\vec{\mathbf{b}} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$.

6. Find all solutions to $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 2 & 4 & 3 & 1 \\ -1 & -2 & -6 & -14 \end{bmatrix} \text{ and } \vec{\mathbf{b}} = \begin{bmatrix} -7 \\ -4 \\ 17 \end{bmatrix}$$

7. Create an example of a matrix A and vector $\vec{\mathbf{b}}$ such that $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ has infinitely many solutions and $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has only the trivial solution.