

THEOREM 7

The Unique Representation Theorem

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be a basis for a vector space V . Then for each \mathbf{x} in V , there exists a unique set of scalars c_1, \dots, c_n such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \dots + c_n\mathbf{b}_n \quad (1)$$

Proof: Since \mathcal{B} spans V , we know such scalars exist.
To show the scalars are unique:

- Suppose $\vec{\mathbf{x}}$ can also be expressed as

$$\vec{\mathbf{x}} = d_1\vec{\mathbf{b}}_1 + \dots + d_n\vec{\mathbf{b}}_n$$

- Then subtracting gives

$$\vec{\mathbf{0}} = (c_1 - d_1)\vec{\mathbf{b}}_1 + \dots + (c_n - d_n)\vec{\mathbf{b}}_n$$

- Since \mathcal{B} is a linearly independent set, we know that

$$c_1 = d_1, c_2 = d_2, \dots, c_n = d_n \quad \square$$

Give the dimension of each vector space

1. \mathbb{R}^5

2. \mathbb{R}^n

3. \mathbb{P}_2

4. \mathbb{P}_n

5. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & -2 & 3 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

(a) $\text{col}(A)$

(b) $\text{nul}(A)$

(c) $\text{row}(A)$

THEOREM 9

If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Proof: Let $\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_p\}$ be a set in V where $p > n$.

Then $\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_p\}$ is linearly dependent if there exists a nontrivial solution to

$$x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + \cdots x_p\vec{\mathbf{v}}_p = \vec{\mathbf{0}}$$

Overview:

- We will convert this into a matrix equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ where A is $n \times p$.
- Since $p > n$, A has a free variable, and there exists a non-trivial solution to the homogeneous system.
- Thus, $\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_p\}$ is a linearly dependent set.
- Note this applies to *any* vector space V , not just \mathbb{R}^n

Since \mathcal{B} is a basis for V , we can write

$$\begin{aligned}a_{11}\vec{\mathbf{b}}_1 + a_{12}\vec{\mathbf{b}}_2 + \cdots + a_{1n}\vec{\mathbf{b}}_n &= \vec{\mathbf{v}}_1 \\a_{21}\vec{\mathbf{b}}_1 + a_{22}\vec{\mathbf{b}}_2 + \cdots + a_{2n}\vec{\mathbf{b}}_n &= \vec{\mathbf{v}}_2 \\&\vdots \\a_{p1}\vec{\mathbf{b}}_1 + a_{p2}\vec{\mathbf{b}}_2 + \cdots + a_{pn}\vec{\mathbf{b}}_n &= \vec{\mathbf{v}}_p\end{aligned}$$

Remember we are looking for a non-trivial solution to

$$x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2 + \cdots + x_p\vec{\mathbf{v}}_p = \vec{\mathbf{0}}$$

which becomes

$$\begin{aligned}&x_1(a_{11}\vec{\mathbf{b}}_1 + a_{12}\vec{\mathbf{b}}_2 + \cdots + a_{1n}\vec{\mathbf{b}}_n) + \\&x_2(a_{21}\vec{\mathbf{b}}_1 + a_{22}\vec{\mathbf{b}}_2 + \cdots + a_{2n}\vec{\mathbf{b}}_n) + \\&\cdots + x_p(a_{p1}\vec{\mathbf{b}}_1 + a_{p2}\vec{\mathbf{b}}_2 + \cdots + a_{pn}\vec{\mathbf{b}}_n) = \vec{\mathbf{0}}\end{aligned}$$

We can rearrange

$$\begin{aligned} & x_1(a_{11}\vec{\mathbf{b}}_1 + a_{12}\vec{\mathbf{b}}_2 + \cdots + a_{1n}\vec{\mathbf{b}}_n) + \\ & x_2(a_{21}\vec{\mathbf{b}}_1 + a_{22}\vec{\mathbf{b}}_2 + \cdots + a_{2n}\vec{\mathbf{b}}_n) + \\ & \cdots + x_p(a_{p1}\vec{\mathbf{b}}_1 + a_{p2}\vec{\mathbf{b}}_2 + \cdots + a_{pn}\vec{\mathbf{b}}_n) = \vec{\mathbf{0}} \end{aligned}$$

to

$$\begin{aligned} & (x_1a_{11} + x_2a_{21} + \cdots + x_pa_{p1})\vec{\mathbf{b}}_1 + \\ & (x_1a_{12} + x_2a_{22} + \cdots + x_pa_{p2})\vec{\mathbf{b}}_2 + \\ & \cdots + (x_1a_{1n} + x_2a_{2n} + \cdots + x_pa_{pn})\vec{\mathbf{b}}_n = \vec{\mathbf{0}} \end{aligned}$$

Remember $\{\vec{\mathbf{b}}_1, \dots, \vec{\mathbf{b}}_n\}$ is a linearly independent set.

Since $\{\vec{\mathbf{b}}_1, \dots, \vec{\mathbf{b}}_n\}$ is linearly independent, we get

$$x_1 a_{11} + x_2 a_{21} + \dots + x_p a_{p1} = 0$$

$$x_1 a_{12} + x_2 a_{22} + \dots + x_p a_{p2} = 0$$

$$\vdots$$

$$x_1 a_{1n} + x_2 a_{2n} + \dots + x_p a_{pn} = 0$$

This converts to the matrix equation

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{p1} \\ a_{12} & a_{22} & \cdots & a_{p2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \vec{\mathbf{0}}$$

which is the same as $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ where A is $n \times p$ with $p > n$.

Thus, A has a free variable and $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$ has a non-trivial solution.

Thus, $\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_p\}$ must be linearly dependent. \square