Diagonalizable:
A square matrix A is **diagonalizable** iff we can write $A = PDP^{-1}$ where D is diagonal.

Theorem 5.5:
A is diagonalizable iff A has n linearly independent eigenvectors.

In particular, if $A = PDP^{-1}$ then the columns in P are n linearly independent eigenvectors of A and the entries in the diagonal of D are the corresponding eigenvalues of A.

Theorem 5.6:
If A is $n \times n$ with n distinct eigenvalues, then A is diagonalizable.

Theorem 7.1:
If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

Orthogonally Diagonalizable
An $n \times n$ matrix A is **orthogonally diagonalizable** iff there is an orthogonal matrix P (so $P^{-1} = P^T$) and a diagonal matrix D such that

$$A = PDP^{-1} = PDP^T$$

Theorem 7.2:
An $n \times n$ matrix A is orthogonally diagonalizable iff A is a symmetric matrix.

Theorem 7.3: The Spectral Theorem for Symmetric Matrices
An $n \times n$ symmetric matrix A has the following properties:

a. A has n real eigenvalues, counting multiplicities.

b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

c. The eigenspaces are mutually orthogonal.

d. A is orthogonally diagonalizable.

Spectral Decomposition for Symmetric Matrices:
If A is an $n \times n$ symmetric matrix with orthonormal eigenvectors $\vec{\mathbf{u_1}}, \vec{\mathbf{u_2}}, \ldots, \vec{\mathbf{u_n}}$ with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the **spectral decomposition** of A is

$$A = \lambda_1 \vec{\mathbf{u_1}} \vec{\mathbf{u_1}}^T + \lambda_2 \vec{\mathbf{u_2}} \vec{\mathbf{u_2}}^T + \cdots + \lambda_n \vec{\mathbf{u_n}} \vec{\mathbf{u_n}}^T$$

Singular Values of an $m \times n$ Matrix:
Let A be an $m \times n$ matrix. Then $A^T A$ is an $n \times n$ symmetric matrix.

The eigenvalues of $A^T A$ are all nonnegative. Reorder so that the eigenvalues are ordered

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$$

The **singular values** of A are the square roots of the eigenvalues of $A^T A$:

$$\sigma_1 = \sqrt{\lambda_1} \quad \geq \quad \sigma_2 = \sqrt{\lambda_2} \quad \geq \quad \cdots \quad \geq \quad \sigma_n = \sqrt{\lambda_n}$$

Singular value decomposition:
Let A be an $m \times n$ matrix with rank r. Then there exists

- an $m \times n$ matrix $\Sigma = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ where D is an $r \times r$ diagonal matrix with the first r singular values of A,

  $$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0,$$

  on its diagonal,
- an $m \times m$ orthogonal matrix U, and
- an $n \times n$ orthogonal matrix V

such that

$$A = U \Sigma V^T$$