Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Then $\vec{\mathbf{x}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector of A

(a) True, and I can explain why

(b) True, but I am unsure why

(c) False, and I can explain why

(d) False, but I am unsure why

(e) Errr. . .

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. Then $\lambda = 3$ is an eigenvalue of A

(a) True, and I can explain why

(b) True, but I am unsure why

(c) False, and I can explain why

(d) False, but I am unsure why

(e) Errr. . .

Let $B = \begin{bmatrix} 75/100 & 15/100 & 5/100 \\ 15/100 & 80/100 & 10/100 \\ 10/100 & 5/100 & 85/100 \end{bmatrix}$

1. For B, find
   (a) The characteristic polynomial
   (b) The eigenvalues
   (c) The corresponding eigenvectors

2. Repeat for ref(B)

Theorem 5.2: If $\vec{v_1}, \ldots, \vec{v_r}$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, then the set $\{\vec{v_1}, \ldots, \vec{v_r}\}$ is linearly independent.

Proof: Suppose $\{\vec{v_1}, \ldots, \vec{v_r}\}$ is a linearly dependent set.
We will show that we get a contradiction.

- Since the vectors are non-zero, we know that at least one must be a linear combination of the others.
- Let $p+1$ be the lowest index of a dependent vector so that $\{\vec{v_1}, \ldots, \vec{v_p}\}$ is linearly independent and

$$c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_p\vec{v_p} = \vec{v_{p+1}} \quad (*)$$

- Multiply both sides by λ_{p+1} (we'll see why in a minute)

$$c_1\lambda_{p+1}\vec{v_1} + c_2\lambda_{p+1}\vec{v_2} + \cdots + c_p\lambda_{p+1}\vec{v_p} = \lambda_{p+1}\vec{v_{p+1}} \quad (**)$$

Theorem 5.2: If $\vec{v_1}, \ldots, \vec{v_r}$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, then the set $\{\vec{v_1}, \ldots, \vec{v_r}\}$ is linearly independent.

$$c_1\vec{\mathbf{v_1}} + c_2\vec{\mathbf{v_2}} + \cdots + c_p\vec{\mathbf{v_p}} = \vec{\mathbf{v_{p+1}}} \quad (*)$$

$$c_1\lambda_{p+1}\vec{\mathbf{v_1}} + c_2\lambda_{p+1}\vec{\mathbf{v_2}} + \cdots + c_p\lambda_{p+1}\vec{\mathbf{v_p}} = \lambda_{p+1}\vec{\mathbf{v_{p+1}}} \quad (**)$$

- Multiply both sides of (*) by A

$$c_1A\vec{\mathbf{v_1}} + c_2A\vec{\mathbf{v_2}} + \cdots + c_pA\vec{\mathbf{v_p}} = A\vec{\mathbf{v_{p+1}}}$$

$$c_1\lambda_1\vec{\mathbf{v_1}} + c_2\lambda_2\vec{\mathbf{v_2}} + \cdots + c_p\lambda_p\vec{\mathbf{v_p}} = \lambda_{p+1}\vec{\mathbf{v_{p+1}}} \quad (***)$$

- Subtract (**) from (***)

$$c_1(\lambda_1 - \lambda_{p+1})\vec{\mathbf{v_1}} + c_2(\lambda_2 - \lambda_{p+1})\vec{\mathbf{v_2}} + \cdots + c_p(\lambda_p - \lambda_{p+1})\vec{\mathbf{v_p}} = \vec{\mathbf{0}}$$

Theorem 5.2: If $\vec{\mathbf{v_1}}, \ldots, \vec{\mathbf{v_r}}$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$, then the set $\{\vec{\mathbf{v_1}}, \ldots, \vec{\mathbf{v_r}}\}$ is linearly independent.

$$c_1(\lambda_1 - \lambda_{p+1})\vec{\mathbf{v_1}} + c_2(\lambda_2 - \lambda_{p+1})\vec{\mathbf{v_2}} + \cdots + c_p(\lambda_p - \lambda_{p+1})\vec{\mathbf{v_p}} = \vec{\mathbf{0}}$$

- But $\lambda_i - \lambda_{p+1} \neq 0$ since the eigenvalues are distinct and $\{\vec{\mathbf{v_1}}, \ldots, \vec{\mathbf{v_p}}\}$ is a linearly independent set, giving that
$$c_1 = c_2 = \cdots = c_p = 0$$
- Since
$$c_1\vec{\mathbf{v_1}} + c_2\vec{\mathbf{v_2}} + \cdots + c_p\vec{\mathbf{v_p}} = \vec{\mathbf{v_{p+1}}} \quad (*)$$
this contradicts that $\vec{\mathbf{v_{p+1}}}$ is non-zero.

Therefore, $\{\vec{\mathbf{v_1}}, \ldots, \vec{\mathbf{v_r}}\}$ are linearly independent. $\square$