Theorem 4.9: If V has basis $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$, then every set in V with more than n vectors is linearly dependent in V.

Proof: Let $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ be a set in V where $p > n$. Then $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is linearly dependent if there exists a nontrivial solution to

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots x_p\mathbf{v_p} = \mathbf{0}$$

Overview:

- We will convert this into a matrix equation $A\mathbf{x} = \mathbf{0}$ where A is $n \times p$.
- Since $p > n$, A has a free variable, and there exists a non-trivial solution to the homogeneous system.
- Thus, $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a linearly dependent set.

Since $\mathcal{B}$ is a basis for V, we can write

$$\begin{aligned}
a_{11}\mathbf{b_1} + a_{12}\mathbf{b_2} + \cdots + a_{1n}\mathbf{b_n} &= \mathbf{v_1} \\
a_{21}\mathbf{b_1} + a_{22}\mathbf{b_2} + \cdots + a_{2n}\mathbf{b_n} &= \mathbf{v_2} \\
&\vdots \\
a_{p1}\mathbf{b_1} + a_{p2}\mathbf{b_2} + \cdots + a_{pn}\mathbf{b_n} &= \mathbf{v_p}
\end{aligned}$$

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$$\begin{aligned} a_{11}\mathbf{b_1} + a_{12}\mathbf{b_2} + \cdots + a_{1n}\mathbf{b_n} &= \mathbf{v_1} \\ a_{21}\mathbf{b_1} + a_{22}\mathbf{b_2} + \cdots + a_{2n}\mathbf{b_n} &= \mathbf{v_2} \\ &\vdots \\ a_{p1}\mathbf{b_1} + a_{p2}\mathbf{b_2} + \cdots + a_{pn}\mathbf{b_n} &= \mathbf{v_p} \end{aligned}$$

Remember we are looking for a non-trivial solution to

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots x_p\mathbf{v_p} = \mathbf{0}$$

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&\vdots \\
a_{p1}\mathbf{b_1} + a_{p2}\mathbf{b_2} + \cdots + a_{pn}\mathbf{b_n} &= \mathbf{v_p}
\end{aligned}$$

Remember we are looking for a non-trivial solution to

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots x_p\mathbf{v_p} = \mathbf{0}$$

which becomes

$$\begin{aligned}
x_1(a_{11}\mathbf{b_1} + a_{12}\mathbf{b_2} + \cdots + a_{1n}\mathbf{b_n}) + \\
x_2(a_{21}\mathbf{b_1} + a_{22}\mathbf{b_2} + \cdots + a_{2n}\mathbf{b_n}) + \\
\cdots + x_p(a_{p1}\mathbf{b_1} + a_{p2}\mathbf{b_2} + \cdots + a_{pn}\mathbf{b_n}) = \mathbf{0}
\end{aligned}$$

We can rearrange

$$\begin{aligned} x_1(a_{11}\mathbf{b_1} + a_{12}\mathbf{b_2} + \cdots + a_{1n}\mathbf{b_n}) + \\ x_2(a_{21}\mathbf{b_1} + a_{22}\mathbf{b_2} + \cdots + a_{2n}\mathbf{b_n}) + \\ \cdots + x_p(a_{p1}\mathbf{b_1} + a_{p2}\mathbf{b_2} + \cdots + a_{pn}\mathbf{b_n}) = \mathbf{0} \end{aligned}$$

to

$$\begin{aligned} (x_1 a_{11} + x_2 a_{21} + \cdots + x_p a_{p1})\mathbf{b_1} + \\ (x_1 a_{12} + x_2 a_{22} + \cdots + x_p a_{p2})\mathbf{b_2} + \\ \cdots + (x_1 a_{1n} + x_2 a_{2n} + \cdots + x_p a_{pn})\mathbf{b_n} = \mathbf{0} \end{aligned}$$

Remember $\{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ is a linearly independent set.

Since $\{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ is a linearly independent, we get

$$\begin{aligned} x_1 a_{11} + x_2 a_{21} + \cdots + x_p a_{p1} &= 0 \\ x_1 a_{12} + x_2 a_{22} + \cdots + x_p a_{p2} &= 0 \\ &\vdots \\ x_1 a_{1n} + x_2 a_{2n} + \cdots + x_p a_{pn} &= 0 \end{aligned}$$

This converts to the matrix equation

$$\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{p1} \\ a_{12} & a_{22} & \cdots & a_{p2} \\ \vdots & & & \\ a_{1n} & a_{2n} & \cdots & a_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \mathbf{0}$$

Which is the same as $A\mathbf{x} = \mathbf{0}$ where A is $n \times p$ with $p > n$.
Thus, A has a free variable and $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.
Thus, $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ must be linearly dependent. $\square$