Thm 5.5: A is diagonalizable iff A has n linearly independent eigenvectors. In particular, if $A = PDP^{-1}$ then the columns in P are n linearly independent eigenvectors of A and the entries in the diagonal of D are the corresponding eigenvalues of A.

($\Rightarrow$) Suppose A is diagonalizable where $A = PDP^{-1}$.

- We need to show A has n linearly independent eigenvectors.
- Let $P = [\mathbf{v_1}, \ldots, \mathbf{v_n}]$ and call the diagonal entries of D be $\lambda_1, \ldots, \lambda_n$.
- We will show λ_i is an eigenvalue with corresponding eigenvector $\mathbf{v_i}$.
- Then

$$\begin{aligned} AP &= PD \\ A[\mathbf{v_1}, \ldots, \mathbf{v_n}] &= [\mathbf{v_1}, \ldots, \mathbf{v_n}]D \\ [A\mathbf{v_1} \ \ldots \ A\mathbf{v_n}] &= [\lambda_1\mathbf{v_1} \ \ldots \ \lambda_n\mathbf{v_n}] \end{aligned}$$

- Thus the columns in P are eigenvectors and the entries in D are eigenvalues of A.
- Since P is invertible, its columns form a linearly independent set.
- Therefore, A has n linearly independent eigenvectors.

($\Leftarrow$) Suppose A has n linearly independent eigenvectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$.

- We need to show that A is diagonalizable.
- Form the matrix $P = [\mathbf{v_1}, \ldots, \mathbf{v_n}]$ and place the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ on the diagonal of a matrix D.
- Then

$$\begin{aligned} AP &= A[\mathbf{v_1}, \ldots, \mathbf{v_n}] \\ &= [A\mathbf{v_1} \ \ldots \ A\mathbf{v_n}] \\ &= [\lambda_1\mathbf{v_1} \ \ldots \ \lambda_n\mathbf{v_n}] \\ &= [\mathbf{v_1}, \ldots, \mathbf{v_n}]D \\ &= PD \end{aligned}$$

- Since the columns of P are linearly independent, P is invertible.
- Thus $A = PDP^{-1}$ and A is diagonalizable. $\square$

Recap

Theorem 5.5: A is diagonalizable iff A has n linearly independent eigenvectors.
In particular, if $A = PDP^{-1}$ then the columns in P are n linearly independent eigenvectors of A and the entries in the diagonal of D are the corresponding eigenvalues of A.

Theorem 5.6: If A is $n \times n$ with n distinct eigenvalues, then A is diagonalizable.

Note: A may be diagonalizable with less than n distinct eigenvalues if its repeated eigenvalues have enough linearly independent eigenvectors.

Classify the origin as a sink (attractor), source (repeller), or saddle for the dynamical system $\mathbf{x_{k+1}} = A\mathbf{x_k}$.

Explain your answer using a plot from Mathematica and the eigenvalues and eigenvectors of A.

1. $A = \frac{1}{96}\begin{bmatrix} 47 & 3 \\ 5 & 33 \end{bmatrix}$
2. $A = \frac{1}{15}\begin{bmatrix} 3 & 6 \\ -14 & 47 \end{bmatrix}$
3. $A = \frac{1}{32}\begin{bmatrix} 89 & -21 \\ -35 & -9 \end{bmatrix}$
4. $A = \begin{bmatrix} 1 & -18 \\ -3 & 4 \end{bmatrix}$
5. $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$
6. $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$