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Proof: Suppose $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ is a linearly dependent set.

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Multiplying both sides of $(*)$ by λ_{p+1} and subtracting from $(**)$ gives

$$c_1(\lambda_1 - \lambda_{p+1})\mathbf{v_1} + c_2(\lambda_2 - \lambda_{p+1})\mathbf{v_2} + \cdots + c_p(\lambda_p - \lambda_{p+1}))\mathbf{v_p} = \mathbf{0}$$

Proof of Theorem 5.2 (continued)

We have

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Proof of Theorem 5.2 (continued)

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Notice that $\lambda_i - \lambda_{p+1} \neq 0$ for all i since the eigenvalues are distinct.

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Thus, we must have $c_1 = c_2 = \cdots = c_p = 0$ since $\{\mathbf{v_1}, \ldots, \mathbf{v_p}\}$ is a linearly independent set.

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Thus $\mathbf{v_{p+1}} = 0$. This contradicts that v_{p+1} is an eigenvector.

Therefore, $\{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ is a linearly independent set. $\square$

1. Let $A = \begin{bmatrix} 1 & -18 \\ -3 & 4 \end{bmatrix}$
   (a) Find the eigenvectors and eigenvalues of A
   (b) Factor A into a product PDP^{-1}.
   (c) Use your factorization to compute A^{20}.

2. Construct a matrix A with eigenvalues $0, 2, 3$ and eigenvectors $(1, 3, -2)$, $(3, 2, 0)$, and $(-2, 1, 4)$, respectively.

3. Is $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$ diagonalizable?

4. True or False
   (a) If A is diagonalizable, then A invertible.
   (b) If A is invertible, then A is diagonalizable.